Fredholm Perturbation of Spectra of 2×2 Upper Triangular Matrix *

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Abstract As we knew, study the perturbation theory of spectra of operator is a very important project in mathematics physics, in particular, in quantum mechanics. In this paper, we characterize the Fredholm perturbation for the Weyl spectrum, essential spectrum, spectrum, left spectrum, right spectrum, lower semi-Fredholm spectrum, upper semi-Weyl spectrum and lower semi-Weyl spectrum of upper triangular operator matrix $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.

Keywords Operator matrix; spectra; perturbation.

AMS classifications 47A10

1 Introduction

Let H and K be the complex infinite dimensional separable Hilbert spaces, $B(H,K)$ be the set of all bounded linear operators from H into K. For simplicity, we write $B(H,H)$ as $B(H)$. If $T \in B(H,K)$, we use $R(T)$ and $N(T)$ to denote the range and kernel of T, respectively, and define $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim(K/R(T))$. For $T \in B(H,K)$, if $R(T)$ is closed and $\alpha(T) < \infty$, we call T an upper semi-Fredholm operator; if $\beta(T) < \infty$, then T is called a lower semi-Fredholm operator. If T is either an upper or lower semi-Fredholm operator, then T is called a semi-Fredholm operator. In this case, the index of T is defined as $\mathrm{ind}(T) = \alpha(T) - \beta(T)$. If T is a semi-Fredholm operator with $\alpha(T) < \infty$ and $\beta(T) < \infty$, then T is called a Fredholm operator. For $T \in B(H)$, the ascent $\mathrm{asc}(T)$ and the descent $\mathrm{des}(T)$ are given by $\mathrm{asc}(T) = \inf\{k \geqslant 0 : N(T^k) = N(T^{k+1})\}$ and $\mathrm{des}(T) = \inf\{k \geqslant 0 : R(T^k) = R(T^{k+1})\}$, respectively; the infimum over the empty set is taken to be ∞.

Let $G(H,K), G_l(H,K)$, $G_r(H,K), \Phi(H,K), \Phi_+(H,K)$ and $\Phi_-(H,K)$, respectively, denote the sets of all invertible operators, left invertible operators, right invertible operators, Fredholm operators, upper semi-Fredholm operators and lower semi-Fredholm operators from H into K. The sets of all Weyl operators, upper semi-Weyl operators and lower semi-Weyl operators from H into K are defined, respectively, by

*This work is supported by the NSF of China (Grant Nos. 10771034, 10771191 and 10471124) and the NSF of Fujian Province of China (Grant Nos. Z0511019, S0650009).

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$\Phi_0(H,K) := \{T \in \Phi(H,K) : \mathrm{ind}(T) = 0\}$,
$\Phi_+^-(H,K) := \{T \in \Phi_+(H,K) : \mathrm{ind}(T) \leq 0\}$,
$\Phi_-^+(H,K) := \{T \in \Phi_-(H,K) : \mathrm{ind}(T) \geq 0\}$.

When $H = K$, the above 9 kind operator classes are also abbreviated as $G(H), G_l(H), G_r(H), \Phi(H)$, $\Phi_+(H)$, $\Phi_-(H)$, $\Phi_0(H)$, $\Phi_+^-(H)$ and $\Phi_-^+(H)$, respectively.

For $T \in B(H)$, its corresponding spectra are, respectively, defined by

the spectrum: $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$,
the left spectrum: $\sigma_l(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left invertible}\}$,
the right spectrum: $\sigma_r(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not right invertible}\}$,
the essential spectrum: $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi(H)\}$,
the upper semi-Fredholm spectrum: $\sigma_{SF+}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_+(H)\}$,
the lower semi-Fredholm spectrum: $\sigma_{SF-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_-(H)\}$,
the Weyl spectrum: $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_0(H)\}$,
the upper semi-Weyl spectrum: $\sigma_{aw}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_+^-(H)\}$,
the lower semi-Weyl spectrum: $\sigma_{sw}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_-^+(H)\}$,
the Browder spectrum: $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_b(H)\}$, where $\Phi_b(H) := \{T \in \Phi(H) : \mathrm{asc}(T) < \infty \text{ and } \mathrm{des}(T) < \infty\}$.

It is well known that all the above spectra are compact nonempty subsets of complex plane $\mathbb{C}$.

Let H be a Hilbert space and T be a bounded linear operator defined on H and H_1 be an invariant closed subspace of T. Then T can be represented by the form of

$$T = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : H_1 \oplus H_1^\perp \to H_1 \oplus H_1^\perp,$$

which motivated the interest in 2×2 upper-triangular operator matrices (see [1-19]).

Henceforth, for $A \in B(H)$, $B \in B(K)$ and $C \in B(K,H)$, we put $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. It is clear that $M_C \in B(H \oplus K)$. Recent, people studied the perturbation theory of some spectra of M_C, for example, in [8], for the spectrum $\sigma(M_C)$, the perturbation result is

$$\bigcap_{C \in B(K,H)} \sigma(M_C) = \sigma_l(A) \cup \sigma_r(B) \cup \{\lambda \in \mathbb{C} : \alpha(B - \lambda) \neq \beta(A - \lambda)\}. \tag{1}$$

In [5], for the Weyl spectrum $\sigma_w(M_C)$ and the essential spectrum $\sigma_e(M_C)$, the perturbation results are

$$\bigcap_{C \in B(K,H)} \sigma_w(M_C) = \sigma_{SF+}(A) \cup \sigma_{SF-}(B) \cup \{\lambda \in \mathbb{C} : \alpha(A-\lambda) + \alpha(B-\lambda) \neq \beta(A-\lambda) + \beta(B-\lambda)\} \tag{2}$$

and

$$\bigcap_{C \in B(K,H)} \sigma_e(M_C) = \sigma_{SF+}(A) \cup \sigma_{SF-}(B) \cup$$
$$\{\lambda \in \mathbb{C} : \min(\beta(A-\lambda), \alpha(B-\lambda)) < \max(\beta(A-\lambda), \alpha(B-\lambda)) = \infty\}. \tag{3}$$

In [1-3, 10], the authors also characterize completely sets $\bigcap_{C \in B(K,H)} \sigma_*(M_C)$, where $\sigma_*(M_C)$ may be the Browder spectrum, left spectrum, right spectrum, lower semi-Fredholm spectrum, upper semi-Fredholm spectrum, lower semi-Weyl spectrum or upper semi-Weyl spectrum of M_C, respectively.

Moreover, in [13-15], for the spectra $\sigma_*(M_C)$, where $\sigma_* = \sigma_r, \sigma_{SF-}$ or σ_{sw}, its perturbation result is

$$\bigcap_{C \in G(K,H)} \sigma_*(M_C) = \Big(\bigcap_{C \in B(K,H)} \sigma_*(M_C)\Big) \cup \{\lambda \in \mathbb{C} : A - \lambda \text{ is compact}\}; \tag{4}$$

for the spectra $\sigma_*(M_C)$, where $\sigma_* = \sigma_l, \sigma_{SF+}$ or σ_{aw}, its perturbation result is

$$\bigcap_{C\in G(K,H)} \sigma_*(M_C) = (\bigcap_{C\in B(K,H)} \sigma_*(M_C)) \cup \{\lambda \in \mathbb{C} : B-\lambda \,\text{is compact}\}; \tag{5}$$

for the spectra $\sigma_*(M_C)$, where $\sigma_* = \sigma, \sigma_e$ or σ_w, its perturbation result is

$$\bigcap_{C\in G(K,H)} \sigma_*(M_C) = (\bigcap_{C\in B(K,H)} \sigma_*(M_C)) \cup \{\lambda \in \mathbb{C} : A-\lambda \,\text{or}\, B-\lambda \,\text{is compact}\}. \tag{6}$$

Note that equations (1) to (3) showed the perturbation of all bounded linear operator C in $B(K,H)$, and equations (4) to (6) showed the perturbation of all bounded invertible linear operator C in $G(K,H)$.

In this paper, we characterize the Fredholm perturbation for the Weyl spectrum, essential spectrum, spectrum, left spectrum, right spectrum, lower semi-Fredholm spectrum, upper semi-Weyl spectrum and lower semi-Weyl spectrum of M_C.

2 Main results and proofs

At first, in order to characterize the perturbation of Weyl spectrum of M_C, we need the following:
Lemma 1. For a given pair $(A,B)\in B(H)\times B(K)$, the following statements are equivalent:

(i). there exists some $C\in B(K,H)$ such that $M_C\in\Phi_0(H\oplus K)$,

(ii). $A\in\Phi_+(H)$, $B\in\Phi_-(K)$ and $\alpha(A)+\alpha(B)=\beta(A)+\beta(B)$,

(iii). there exists some $Q\in G(K,H)$ such that $M_Q\in\Phi_0(H\oplus K)$,

(iv). there exists some $Q\in\Phi(K,H)$ such that $M_Q\in\Phi_0(H\oplus K)$.

Proof. (i)⇔(ii) was proved in [5, Theorem 3.6].

(ii)⇒(iii). It is sufficient to prove that if $A\in\Phi_+(H)$, $B\in\Phi_-(K)$ and $\beta(A)=\alpha(B)=\infty$, then there exists $Q\in G(K,H)$ such that $M_Q\in\Phi_0(H\oplus K)$. To show this, there are three cases to consider:

Case 1. Suppose $\alpha(A)=\beta(B)<\infty$. Define an operator $Q: K\to H$ by $Q=\begin{pmatrix} T_1 & 0 \\ 0 & T_2\end{pmatrix}$: $N(B)\oplus N(B)^\perp \to R(A)^\perp\oplus R(A)$, where T_1 and T_2 are invertible operators. Obviously, $Q\in G(K,H)$ and $M_Q\in\Phi(H\oplus K)$. Also, it is evident that $N(M_Q)=N(A)\oplus\{0\}$ and $R(M_Q)^\perp=\{0\}\oplus R(B)^\perp$. Thus $\alpha(M_Q)=\beta(M_Q)=\alpha(A)=\beta(B)<\infty$, and hence $M_Q\in\Phi_0(H\oplus K)$ is clear.

Case 2. Suppose $\beta(B)<\alpha(A)<\infty$ and put $l=\alpha(A)-\beta(B)$. Note that $\beta(A)=\dim N(B)^\perp=\infty$, let $R(A)^\perp=H_1\oplus H_2$ and $\dim H_2=l$, $N(B)^\perp=K_1\oplus K_2$ and $\dim(K_1)=l$.

Define an operator $Q: K\to H$ by $Q=\begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3\end{pmatrix}$: $N(B)\oplus K_1\oplus K_2\to H_1\oplus H_2\oplus R(A)$, where T_1, T_2 and T_3 are invertible operators. Obviously, $Q\in B(K,H)$ is invertible. Now we claim that $M_Q\in\Phi_0(H\oplus K)$. In fact, M_Q has the following form: $M_Q=\begin{pmatrix} 0 & 0 & T_1 & 0 & 0 \\ 0 & 0 & 0 & T_2 & 0 \\ 0 & A_1 & 0 & 0 & T_3 \\ 0 & 0 & 0 & B_1 & B_2 \\ 0 & 0 & 0 & 0 & 0\end{pmatrix}$: $N(A)\oplus N(A)^\perp\oplus N(B)\oplus K_1\oplus K_2 \longrightarrow H_1\oplus H_2\oplus R(A)\oplus R(B)\oplus R(B)^\perp$, where $A_1\in B(N(B)^\perp, R(A))$ and $(B_1\ B_2)\in B((K_1\oplus K_2), R(B))$ are invertible operators. Moreover, observe that $\dim K_1<\infty$, we have $B_1\in G(K_1, R(B_1))$, $B_2\in G(K_2, R(B_2))$ and $\dim K_1=\dim R(B_1)=\dim(R(B)\ominus R(B_2))$.

Now let $W_1 = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & -B_1T_2^{-1} & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix} : N(A) \oplus N(A)^{\perp} \oplus N(B) \oplus K_1 \oplus K_2 \longrightarrow N(A) \oplus N(A)^{\perp} \oplus N(B) \oplus K_1 \oplus K_2$,

and $W_2 = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & -A_1^{-1}T_3 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix} : H_1 \oplus H_2 \oplus R(A) \oplus R(B) \oplus R(B)^{\perp} \longrightarrow H_1 \oplus H_2 \oplus R(A) \oplus R(B) \oplus R(B)^{\perp}$.

Then $W_1M_QW_2 = \begin{pmatrix} 0 & 0 & T_1 & 0 & 0 \\ 0 & 0 & 0 & T_2 & 0 \\ 0 & A_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} : N(A) \oplus N(A)^{\perp} \oplus N(B) \oplus K_1 \oplus K_2 \longrightarrow H_1 \oplus H_2 \oplus R(A) \oplus R(B) \oplus R(B)^{\perp}$. Since A_1, T_1 and T_2 are invertible, we get that $R(W_1M_QW_2) = H_1 \oplus H_2 \oplus R(A) \oplus R(B_2) \oplus \{0\}$ and $N(W_1M_QW_2) = N(A) \oplus \{0\} \oplus \{0\} \oplus \{0\} \oplus \{0\}$, and $R(W_1M_QW_2)^{\perp} = \{0\} \oplus \{0\} \oplus \{0\} \oplus (R(B) \ominus R(B_2)) \oplus R(B)^{\perp}$. Thus $W_1M_QW_2 \in \Phi(H \oplus K)$ and

$$\begin{aligned} \alpha(W_1M_QW_2) &= \alpha(A) = l + \beta(B) \\ &= \dim K_1 + \beta(B) \\ &= \dim(R(B) \ominus R(B_2)) + \beta(B) \\ &= \beta(W_1M_QW_2) < \infty. \end{aligned}$$

So $W_1M_QW_2 \in \Phi_0(H \oplus K)$. Also since W_1 and W_2 are invertible, it follows that $M_Q \in \Phi_0(H \oplus K)$.

Case 3. Suppose $\alpha(A) < \beta(B) < \infty$, put $l = \beta(B) - \alpha(A)$. Since $\dim R(A) = \dim N(B) = \infty$, let R(A)$= H_1 \oplus H_2$ and $\dim H_1 = l$, $N(B) = K_1 \oplus K_2$ and $\dim(K_2) = l$. That $\dim H_2 = \dim(K_1) = \infty$ is clear. Define an operator $Q : K \to H$ by $Q = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix} : K_1 \oplus K_2 \oplus N(B)^{\perp} \to R(A)^{\perp} \oplus H_1 \oplus H_2$, where T_1, T_2 and T_3 are invertible operators. Obviously, $Q \in G(K, H)$. Similar to the proof of Case 2, we can also show that $M_Q \in \Phi_0(H \oplus K)$.

It follows from Case 1 to Case 3 that (ii)$\Rightarrow$(iii).

Finally, (iii)$\Rightarrow$(iv) and (iv)$\Rightarrow$(i) are clear. The lemma is proved.

From Lemma 1 and Equation (2), we have the following:

Theorem 1. For a given pair $(A, B) \in B(H) \times B(K)$, we have

$$\bigcap_{C \in \Phi(K,H)} \sigma_w(M_C) = \bigcap_{C \in G(K,H)} \sigma_w(M_C) = \bigcap_{C \in B(K,H)} \sigma_w(M_C)$$

$$= \sigma_{SF+}(A) \cup \sigma_{SF-}(B) \cup \{\lambda \in \mathbb{C} : \alpha(A - \lambda) + \alpha(B - \lambda) \neq \beta(A - \lambda) + \beta(B - \lambda)\}.$$

In order to characterize the perturbation of essential spectrum of M_C, we need the following:

Lemma 2. For a given pair $(A, B) \in B(H) \times B(K)$, the following statements are equivalent:

(i). there exists some $C \in B(K, H)$ such that $M_C \in \Phi(H \oplus K)$,

(ii). $\begin{cases} A \in \Phi(H) \text{ and } B \in \Phi(K) \\ \text{or } A \in \Phi_+(H), B \in \Phi_-(K) \text{ and } \beta(A) = \alpha(B) = \infty, \end{cases}$

(iii). there exists some $Q \in G(K, H)$ such that $M_Q \in \Phi(H \oplus K)$,

(iv). there exists some $Q \in \Phi(K, H)$ such that $M_Q \in \Phi(H \oplus K)$.

Proof. (i)⇒(ii). Suppose that $M_C \in \Phi(H \oplus K)$ for some $C \in B(K, H)$. It follows from [5, Theorem 3.2] that $A \in \Phi_+(H), B \in \Phi_-(K)$. Moreover, by [19, Lemma 2.2] we have that either both A and B are Fredholm operators or neither A nor B is a Fredholm operator. Thus $\beta(A) = \alpha(B) = \infty$ when neither A nor B is a Fredholm operator.

(ii)⇒(iii). To do this, if $A \in \Phi(H)$ and $B \in \Phi(K)$, then $M_C \in \Phi(H \oplus K)$ for every $C \in B(K, H)$. On the other hand, if $A \in \Phi_+(H)$, $B \in \Phi_-(K)$ and $\beta(A) = \alpha(B) = \infty$. Define an operator $Q : K \to H$ by $Q = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} : N(B) \oplus N(B)^\perp \to R(A)^\perp \oplus R(A)$, where T_1 and T_2 are invertible operators. Obviously, $Q \in G(K, H)$, and it is easy to show that $M_Q \in \Phi(H \oplus K)$.

(iii) ⇒ (iv) and (iv) ⇒ (i) are obvious. The lemma is proved.

From Lemma 2 and Equation (3) we have the following immediately:

Theorem 2. For a given pair $(A, B) \in B(H) \times B(K)$, we have

$$\bigcap_{C \in \Phi(K,H)} \sigma_e(M_C) = \bigcap_{C \in G(K,H)} \sigma_e(M_C) = \bigcap_{C \in B(K,H)} \sigma_e(M_C)$$

$$= \sigma_{SF+}(A) \cup \sigma_{SF-}(B) \cup \{\lambda \in \mathbb{C} : \min(\beta(A - \lambda), \alpha(B - \lambda)) < \max(\beta(A - \lambda), \alpha(B - \lambda)) = \infty\}.$$

In order to characterize the perturbation of spectrum of M_C, we need the following lemma which is a generalization in [9, Theorem 2] in the case of Hilbert spaces:

Lemma 3. For a given pair $(A, B) \in B(H) \times B(K)$, the following statements are equivalent:

(i). there exists some $C \in B(K, H)$ such that M_C is invertible,

(ii). A is left invertible, B is right invertible and $\beta(A) = \alpha(B)$,

(iii). there exists some $Q \in G(K, H)$ such that M_Q is invertible,

(iv). there exists some $Q \in \Phi(K, H)$ such that M_Q is invertible.

Proof. (i)⇒(ii) is prove in [9, Theorem 2]. In fact, if M_C is invertible, it is easy to show that A is left invertible and B is right invertible, which implies that $\alpha(A) = \beta(B) = 0$. Moreover, it follows from Lemma 1 that $\alpha(A) + \alpha(B) = \beta(A) + \beta(B)$, thus $\beta(A) = \alpha(B)$.

(ii)⇒(iii). Suppose A is left invertible, B is right invertible and $\beta(A) = \alpha(B)$. Define an operator $Q : K \to H$ by $Q = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} : N(B) \oplus N(B)^\perp \to R(A)^\perp \oplus R(A)$, where T_1 and T_2 are invertible operators. it is evident that $Q \in G(K, H)$ and $M_Q \in G(H \oplus K)$.

(iii) ⇒ (iv) and (iv) ⇒ (i) are obvious. The lemma is proved.

From Lemma 3 and Equation (1), the following theorem is immediate:

Theorem 3. For a given pair $(A, B) \in B(H) \times B(K)$, We have

$$\bigcap_{C \in \Phi(K,H)} \sigma(M_C) = \bigcap_{C \in G(K,H)} \sigma(M_C) = \bigcap_{C \in B(K,H)} \sigma(M_C)$$

$$= \sigma_l(A) \cup \sigma_r(B) \cup \{\lambda \in \mathbb{C} : \alpha(B - \lambda) \neq \beta(A - \lambda)\}.$$

In order to characterize the perturbation for left spectrum, right spectrum, lower semi-Weyl spectrum, upper semi-Weyl spectrum and lower semi-Fredholm spectrum of M_C, we need the following three lemmas:

Lemma 4. For a given pair $(A,B) \in B(H) \times B(K)$, if either A or B is a compact operator, then for each $C \in \Phi(K,H)$, M_C is not a semi-Fredholm operator.

Proof. If B is a compact operator, then we can claim that M_C is not a semi-Fredholm operator for each $C \in \Phi(K,H)$. If not, assume that $C_0 \in \Phi(K,H)$ such that M_{C_0} is a semi-Fredholm operator. Since $C_0 \in \Phi(K,H)$, there exists $C_1 \in \Phi(H,K)$ such that $C_0C_1 = I + K$, where $K \in B(H)$ is a compact operator. Note that

$$\begin{pmatrix} A & C_0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & 0 \\ -C_1A & I \end{pmatrix} = \begin{pmatrix} A - C_0C_1A & C_0 \\ -BC_1A & B \end{pmatrix} = \begin{pmatrix} -KA & C_0 \\ -BC_1A & B \end{pmatrix},$$

we have that $\begin{pmatrix} -KA & C_0 \\ -BC_1A & B \end{pmatrix}$ is a semi-Fredholm operator. Also since K and B are compact operators, both $\begin{pmatrix} 0 & 0 \\ -BC_1A & 0 \end{pmatrix}$ and $\begin{pmatrix} -KA & 0 \\ 0 & B \end{pmatrix}$ are also compact. Thus $\begin{pmatrix} 0 & C_0 \\ 0 & 0 \end{pmatrix}$ is a semi-Fredholm operator, which is impossible. So M_C is not a semi-Fredholm operator for each $C \in \Phi(K,H)$.

Similarly, we can prove when A is a compact operator, M_C is not a semi-Fredholm operator for each $C \in \Phi(K,H)$. The lemma is proved.

Lemma 5. The following statements are equivalent:

(i). B is not compact,

(ii). for each given $A \in \Phi_+(H)$, if $\beta(A) = \infty$, then there exists an operator $C \in G(K,H)$ such that M_C is an upper semi-Weyl operator and $\alpha(M_C) = \alpha(A)$,

(iii). for each given $A \in \Phi_+(H)$, if $\beta(A) = \infty$, then there exists an operator $C \in \Phi(K,H)$ such that M_C is an upper semi-Weyl operator and $\alpha(M_C) = \alpha(A)$,

(iv). for each given $A \in \Phi_+(H)$, if $\beta(A) = \infty$, then there exists an operator $C \in G(K,H)$ such that M_C is an upper semi-Weyl operator,

(v). for each given $A \in \Phi_+(H)$, if $\beta(A) = \infty$, then there exists an operator $C \in \Phi(K,H)$ such that M_C is an upper semi-Weyl operator,

(vi). for each given $A \in \Phi_+(H)$, if h $\beta(A) = \infty$, then there exists an operator $C \in G(K,H)$ such that M_C is an upper semi-Fredholm operator,

(vii). for each given $A \in \Phi_+(H)$, if $\beta(A) = \infty$, then there exists an operator $C \in \Phi(K,H)$ such that M_C is an upper semi-Fredholm operator.

Proof. Obviously, we only need to prove the implications (i) $\Rightarrow$ (ii) and (vii) $\Rightarrow$ (i).

(vii) $\Rightarrow$ (i). If B is compact, then it follows from Lemma 4 that M_C is not a semi-Fredholm operator for each $C \in \Phi(K,H)$, which contradicts with (vii). Thus B is not compact.

(i) $\Rightarrow$ (ii). Suppose that B is not compact. Then we consider the following two cases:

Case 1. Assume that $R(B)$ is closed. Since the assumption that B is not compact, we have that $\dim N(B)^\perp = \infty$. Also since $\beta(A) = \infty$, let $R(A)^\perp = H_1 \oplus H_2$ with $\dim H_1 = \dim N(B)$ and $\dim H_2 = \infty$. Define an operator $C : K \to H$ by

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} : N(B) \oplus N(B)^\perp \longrightarrow H_1 \oplus (H_2 \oplus R(A)),$$

where $C_1 \in B(N(B), H_1)$ and $C_2 \in B(N(B)^\perp, H_2 \oplus R(A))$ are invertible operators. Obviously, operator C is invertible. By [12, Lemma 2], M_C is an upper semi-Fredholm operator. Moreover, it is easy to prove that $N(M_C) = N(A) \oplus \{0\}$ and $\dim R(M_C)^\perp \geq \dim H_2 = \infty$. Thus, M_C is an upper semi-Weyl operator and $\alpha(M_C) = \alpha(A)$.

Case 2. Assume that $R(B)$ is not closed. By [13, Lemma 3.6] and its proof, we can obtain an operator $C \in G(K,H)$ such that M_C is an upper semi-Weyl operator and $\alpha(M_C) = \alpha(A)$. The lemma is proved.

Duality, we have:

Lemma 6. The following statements are equivalent:

(i). A is not compact,

(ii). for each given $B \in \Phi_-(K)$, if $\alpha(B) = \infty$, then there exists an operator $C \in G(K, H)$ such that M_C is a lower semi-Weyl operator and $\beta(M_C) = \beta(B)$,

(iii). for each given $B \in \Phi_-(K)$, if $\alpha(B) = \infty$, then there exists an operator $C \in \Phi(K, H)$ such that M_C is a lower semi-Weyl operator and $\beta(M_C) = \beta(B)$,

(iv). for each given $B \in \Phi_-(K)$, if $\alpha(B) = \infty$, then there exists an operator $C \in G(K, H)$ such that M_C is a lower semi-Weyl operator,

(v). for each given $B \in \Phi_-(K)$, if $\alpha(B) = \infty$, then there exists an operator $C \in \Phi(K, H)$ such that M_C is a lower semi-Weyl operator,

(vi). for each given $B \in \Phi_-(K)$, if $\alpha(B) = \infty$, then there exists an operator $C \in G(K, H)$ such that M_C is a lower semi-Fredholm operator,

(vii). for each given $B \in \Phi_-(K)$, if $\alpha(B) = \infty$, then there exists an operator $C \in \Phi(K, H)$ such that M_C is a lower semi-Fredholm operator.

Our Theorem 4 and Theorem 5 following show the similar conclusions as Equation (4)-(5).

Theorem 4. For a given pair $(A, B) \in B(H) \times B(K)$, we have

$$\bigcap_{C\in\Phi(K,H)} \sigma_*(M_C) = (\bigcap_{C\in B(K,H)} \sigma_*(M_C)) \cup \{\lambda \in \mathbb{C} : A - \lambda \text{ is compact}\},$$

where $\sigma_* \in \{\sigma_r, \sigma_{SF-}, \sigma_{sw}\}$.

Proof. According to Lemma 4, it is clear that

$$\bigcap_{C\in\Phi(K,H)} \sigma_*(M_C) \supseteq (\bigcap_{C\in B(K,H)} \sigma_*(M_C)) \cup \{\lambda \in \mathbb{C} : A - \lambda \text{ is compact}\}.$$

In order to show the theorem, we only need to prove that

$$\bigcap_{C\in\Phi(K,H)} \sigma_*(M_C) \subseteq (\bigcap_{C\in B(K,H)} \sigma_*(M_C)) \cup \{\lambda \in \mathbb{C} : A - \lambda \text{ is compact}\}.$$

(i). Suppose that $\sigma_*(\cdot) = \sigma_{SF-}(\cdot)$ and $\lambda \notin (\bigcap_{C\in B(K,H)} \sigma_{SF-}(M_C)) \cup \{\lambda \in \mathbb{C} : A-\lambda \text{ is compact}\}$. Then $A - \lambda$ is not compact and there exists $C \in B(K, H)$ such that $M_C - \lambda \in \Phi_-(H \oplus K)$, and hence $B - \lambda \in \Phi_-(K)$.

Case 1. $\alpha(B - \lambda) = \infty$. It follows from Lemma 6 that there exists $C \in \Phi(K, H)$ such that $M_C - \lambda$ is a lower semi-Fredholm operator. This implies that $\lambda \notin \bigcap_{C\in\Phi(K,H)} \sigma_{SF-}(M_C)$. It is clear that

$$\bigcap_{C\in\Phi(K,H)} \sigma_{SF-}(M_C) \subseteq (\bigcap_{C\in B(K,H)} \sigma_{SF-}(M_C)) \cup \{\lambda \in \mathbb{C} : A - \lambda \text{ is compact}\}.$$

Case 2. $\alpha(B - \lambda) < \infty$. This implies that $B - \lambda \in \Phi(K)$, and so $A - \lambda \in \Phi_-(H)$ since $M_C - \lambda \in \Phi_-(H \oplus K)$. Thus, we have that $M_C - \lambda$ is a lower semi-Fredholm operator for each $C \in B(K, H)$, which means $\lambda \notin \bigcap_{C\in\Phi(K,H)} \sigma_{SF-}(M_C)$. Thus

$$\bigcap_{C\in\Phi(K,H)} \sigma_{SF-}(M_C) \subseteq \bigcap_{C\in B(K,H)} \sigma_{SF-}(M_C) \cup \{\lambda \in \mathbb{C} : A - \lambda \text{ is compact}\}.$$

Together Case 1 with Case 2, we have

$$\bigcap_{C\in\Phi(K,H)} \sigma_{SF-}(M_C) = (\bigcap_{C\in B(K,H)} \sigma_{SF-}(M_C)) \cup \{\lambda \in \mathbb{C} : A - \lambda \text{ is compact}\}.$$

(ii). Suppose that $\sigma_*(\cdot) = \sigma_r(\cdot)$ and $\lambda \notin (\bigcap_{C\in B(K,H)} \sigma_r(M_C)) \cup \{\lambda \in \mathbb{C} : A - \lambda \text{ is compact}\}$. Then $A - \lambda$ is not compact and there exists $C \in B(K, H)$ such that $M_C - \lambda \in G_r(H \oplus K)$, and hence $B - \lambda \in G_r(K)$.

Case 1. $\alpha(B - \lambda) = \infty$. It follows from Lemma 6 that there exists $C \in \Phi(K, H)$ such that $M_C - \lambda$ is a lower semi-Weyl operator and $\beta(M_C - \lambda) = \beta(B - \lambda)$. Note that $B - \lambda$ is surjective, then $M_C - \lambda$ is also surjective. This implies that $\lambda \notin \bigcap_{C\in\Phi(K,H)} \sigma_r(M_C)$. It is clear that

$$\bigcap_{C\in\Phi(K,H)} \sigma_r(M_C) \subseteq \bigcap_{C\in B(K,H)} \sigma_r(M_C) \cup \{\lambda \in \mathbb{C} : A - \lambda \text{ is compact}\}.$$

Case 2. $\alpha(B-\lambda) < \infty$. This means that $B-\lambda \in \Phi(K)$, so it is easy to prove that $A-\lambda \in \Phi_-(H)$. Moreover, it follows from [10, Corollary 2] that $\alpha(B - \lambda) \geq \beta(A - \lambda)$. Next we claim that there exists some $C \in \Phi(K, H)$ such that $\lambda \notin \bigcap_{C\in\Phi(K,H)} \sigma_r(M_C)$. For this, let $N(B - \lambda)^\perp = K_1 \oplus K_2$ with $\dim K_2 = \dim \beta(A - \lambda)$. Define an operator $Q : K \to H$ by

$$Q = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} : (N(B-\lambda) \oplus K_1) \oplus K_2 \longrightarrow R(A-\lambda) \oplus R(A-\lambda)^\perp,$$

where $C_1 \in B(N(B-\lambda) \oplus K_1, R(A-\lambda))$ and $C_2 \in B(K_2, R(A-\lambda)^\perp)$ are invertible operators. Obviously, operator $Q \in G(K, H)$ and $M_C - \lambda$ is surjective. Thus

$$\bigcap_{C\in G(K,H)} \sigma_r(M_C) \subseteq \bigcap_{C\in B(K,H)} \sigma_r(M_C) \cup \{\lambda \in \mathbb{C} : A - \lambda \text{ is compact}\}.$$

Together Case 1 with Case 2, we have

$$\bigcap_{C\in\Phi(K,H)} \sigma_r(M_C) = \bigcap_{C\in B(K,H)} \sigma_r(M_C) \cup \{\lambda \in \mathbb{C} : A - \lambda \text{ is compact}\}.$$

Similarly, when $\sigma_* = \sigma_{sw}$, we can prove the conclusion is also true.

By the proof methods of Theorem 4, we can prove the following result:

Theorem 5. For a given pair $(A, B) \in B(H) \times B(K)$, we have

$$\bigcap_{C\in\Phi(K,H)} \sigma_*(M_C) = (\bigcap_{C\in B(K,H)} \sigma_*(M_C)) \cup \{\lambda \in \mathbb{C} : B - \lambda \text{is compact}\},$$

where $\sigma_* \in \{\sigma_l, \sigma_{aw}\}$.

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